YOURSELF: PROVE THAT CHAR. POLY. OF

$$ay'' + by' + cy = 0$$ ← LINEAR HOMOGENEOUS CONSTANT COEF'S

IS $ar^2 + br + c = 0$

SOLVE $y'' + 4y' + 4y = 0$ GUESS $y = e^{rx}$

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0$$

$$r = -2, -2$$

$$y_1 = e^{-2x}, \quad y_2 = e^{-2x}$$

$$y = c_1 e^{-2x} + c_2 e^{-2x} = (c_1 + c_2) e^{-2x} = Ce^{-2x}$$

IS <u>NOT</u> THE GENERAL SOL'N

$$W[e^{-2x}, e^{-2x}] = \begin{vmatrix} e^{-2x} & e^{-2x} \\ -2e^{-2x} & -2e^{-2x} \end{vmatrix} = e^{-2x} \cdot (-2e^{-2x}) - (-2e^{-2x}) e^{-2x} = 0$$

GUESS $y = v(x)e^{-2x} = ve^{-2x}$

$$y' = v'e^{-2x} - 2ve^{-2x}$$

$$y'' = v''e^{-2x} - 2v'e^{-2x} - 2v'e^{-2x} + 4ve^{-2x}$$

$$= v''e^{-2x} - 4v'e^{-2x} + 4ve^{-2x}$$

$$y'' + 4y' + 4y = e^{-2x}(v'' \cancel{-4v'} \cancel{+4v} \cancel{+4v'} \cancel{-8v} \cancel{+4v})$$

$$= v''\underbrace{e^{-2x}}_{\neq 0} = 0$$

$$v'' = 0$$

$$v' = c_1$$

$$v = c_1 x + c_2$$

$y_1 = e^{-2x}$ $\quad y_2 = (c_1 x + c_2)e^{-2x}$

$$= \cancel{c_1} xe^{-2x} \cancel{+ c_2 e^{-2x}} \longrightarrow y_2 = xe^{-2x}$$

YOU: CHECK THIS IS A SOL'N OF $y'' + 4y' + 4y = 0$

$$y = Ae^{-2x} + B(c_1 xe^{-2x} + c_2 e^{-2x})$$

$$= (A + Bc_2)e^{-2x} + Bc_1 xe^{-2x} = k_1 e^{-2x} + k_2 xe^{-2x}$$

$$W\left[e^{-2x}, x e^{-2x}\right]$$

$$= \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & 1 \cdot e^{-2x} - 2xe^{-2x} \end{vmatrix}$$

$$= e^{-4x} - \cancel{2xe^{-4x}} - \cancel{-2xe^{-4x}}$$

$$= e^{-4x} \neq 0$$

SO $y = c_1 e^{-2x} + c_2 x e^{-2x}$ IS THE GENERAL SOL'N OF $y'' + 4y' + 4y = 0$

RENAMED k_1, k_2

IF THE CHARACTERISTIC POLYNOMIAL

1. HAS 2 REAL ROOTS r_1, r_2 $(r_1 \neq r_2)$

CHAR POLY $(r-r_1)(r-r_2)$

$y_1 = e^{r_1 x}$ $y_2 = e^{r_2 x}$ $W[e^{r_1 x}, e^{r_2 x}] \neq 0$

$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ IS THE GENERAL SOL'N

2. HAS A DOUBLE REAL ROOT r_1, r_1 $W[e^{r_1 x}, e^{r_1 x}] = 0$

~~$y = c_1 e^{r_1 x} + c_2 e^{r_1 x}$~~ IS **NOT** THE GENERAL SOL'N

CHAR POLY

$(r-r_1)(r-r_1)$

$= r^2 - 2r_1 r + r_1^2$

$y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$ **IS** THE GENERAL SOL'N

IE. DE WAS $y'' - 2r_1 y' + r_1^2 y = 0$

REPLACE r_1 WITH a (EASIER TO WORK WITH)

$y'' - 2ay' + a^2 y = 0$

$y_1 = e^{ax}$ GUESS $y_2 = V(x)e^{ax} = Ve^{ax}$

$$y = ve^{ax}$$
$$y' = v'e^{ax} + ave^{ax}$$
$$y'' = v''e^{ax} + av'e^{ax} + av'e^{ax} + a^2ve^{ax}$$
$$= v''e^{ax} + 2av'e^{ax} + a^2ve^{ax}$$

$$y'' - 2ay' + a^2y$$
$$= e^{ax}\left(v'' + \cancel{2av'} + \cancel{a^2v} - \cancel{2av'} - \cancel{2a^2v} + \cancel{a^2v}\right)$$

$$= v''\underbrace{e^{ax}}_{\neq 0} = 0$$

$$v'' = 0$$
$$v' = C_1$$
$$v = C_1x + C_2$$
$$y_2 = (C_1x + C_2)e^{ax}$$
$$y_2 = \cancel{C_1}xe^{ax} + \underbrace{\cancel{C_2e^{ax}}}_{\text{PART OF } y_1} \longrightarrow y_2 = xe^{ax}$$

CHECK THAT xe^{ax} IS A SOL'N OF $y'' - 2ay' + a^2y = 0$

$$\longrightarrow y_1 = e^{ax} \quad y_2 = xe^{ax}$$

$$W[e^{ax}, xe^{ax}]$$
$$= \begin{vmatrix} e^{ax} & xe^{ax} \\ ae^{ax} & e^{ax} + axe^{ax} \end{vmatrix}$$
$$= e^{2ax} + \cancel{axe^{2ax}} - \cancel{axe^{2ax}}$$
$$= e^{2ax} \neq 0$$

$$y = C_1e^{ax} + C_2xe^{ax}$$

IS THE GENERAL SOL'N

SOLVE

$$4y''+12y'+9y=0$$

$$\boxed{4r^2+12r+9=0}$$

$$(2r+3)^2=0$$

$$\boxed{r=-\frac{3}{2},\ -\frac{3}{2}}$$

$$\boxed{y=c_1e^{-\frac{3}{2}x}+c_2xe^{-\frac{3}{2}x}}$$

(3) 2 COMPLEX ROOTS $r=a\pm bi$
CONJUGATE

$$y=c_1e^{(a+bi)x}+c_2e^{(a-bi)x}$$

$$y=c_1e^{ax+ibx}+c_2e^{ax-ibx}$$

$$=c_1e^{ax}e^{ibx}+c_2e^{ax}e^{-ibx}$$

UNACCEPTABLE -
COEF'S + INPUTS
WERE ALL REAL
SO, SOL'N SHOULD
BE REAL TOO

WHAT IS $e^{i\theta}$?

<u>MATH IC:</u>

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots$$

$$= 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \dots$$

$$= 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \dots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right)$$

$$+ i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right)$$

$$= \cos\theta + i\sin\theta$$

$i = \sqrt{-1}$

$i^2 = -1$

$i^3 = i^2 \cdot i = -i$

$i^4 = i^3 \cdot i = -i \cdot i$

$= -i^2$

$= 1$

$i^5 = i^4 \cdot i = 1 \cdot i = i$

$i^0 = 1 = i^4 = i^8 \leftarrow$ $i^{4n} = 1$

$i^1 = i = i^5 = i^9$ $\quad i^{4n+1} = i$

$i^2 = -1 = i^6 = i^{10}$ $\quad i^{4n+2} = -1$

$i^3 = -i = i^7 = i^{11}$ $\quad i^{4n+3} = -i$

$$y = c_1 e^{ax} e^{ibx} + c_2 e^{ax} e^{-ibx}$$

$$= c_1 e^{ax}(\cos bx + i \sin bx) + c_2 e^{ax}(\underbrace{\cos(-bx)}_{\cos bx} + i \underbrace{\sin(-bx)}_{-\sin bx})$$

$$= (c_1 + c_2) e^{ax} \cos bx + (c_1 i - c_2 i) e^{ax} \sin bx$$

$$= \underbrace{(c_1 + c_2)} e^{ax} \cos bx + \underbrace{i(c_1 - c_2)} e^{ax} \sin bx$$

$$y = k_1 e^{ax} \cos bx + k_2 e^{ax} \sin bx \quad ?$$

WHAT WAS THE DE?

TEST IF BOTH TERMS ARE SOL'NS

CHECK IF $W \neq 0$